Analytical Dark Solitary Wave Solutions for the Higher Order Nonlinear Schrödinger Equation with Cubic-quintic Terms

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By means of the coupled amplitude-phase method we find analytical dark solitary wave solutions for the higher order nonlinear Schrödinger equation with cubic-quintic terms describing the effects of quintic nonlinearity on the ultra-short (femtosecond) optical soliton propagation in non-Kerr media. The dark solitary wave solution exists even for the coefficients of quintic terms much larger than those of cubic terms.

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Optical solitons in Kerr nonlinear media have been the subject of intense current research motivated by their important applications to high-capacity fiber telecommunications and to all optical switches due to their capability of propagating over long distances without attenuation and changing their shapes [1, 2]. The dynamics of solitons in Kerr media are in general described by the nonlinear Schrödinger (NLS) family of equations with cubic nonlinear terms [3 - 7]. However, as one increases the intensity of the incident light to produce shorter (femtosecond) pulses, non-Kerr nonlinearity effects become important and the dynamics of pulses should be described by the NSL family of equations with higher-order nonlinear terms. As such a model, Radhakrshna, Kundu, and Lakshmanan [8] recently proposed the higher order nonlinear Schrödinger (HONLS) equation with cubic-quintic nonlinear terms arising in non-Kerr media. They investigated an integrable system of coupled HONLS equations with some simplifications in the model parameters and found Lax pair, conserved quantities, exact soliton solutions, and analyzed the two-soliton interaction. However, the model [8] along with the general HONLS models proposed in the literature [9 - 13], is not completely integrable and can not be solved exactly by the inverse scattering method.

To date, exact analytical fundamental solitary wave solutions of the proposed model [8] have not been found. However, analytical and numerical solitary wave solutions have been obtained in the HONLS models in which the effects of quintic nonlinear terms are smaller than the cubic terms [9 - 13]. In this work, by using the coupled amplitude-phase formulation [14, 15], we find analytical fundamental dark solitary wave solutions and the constraint equations for the model coefficients. The results show that the dark solitary wave solution exists even though the coefficients of quintic terms are much larger than those of cubic terms.

The HONLS equation with cubic-quintic nonlinear non-Kerr terms describing the propagation of femtosecond pulses can be written in the form [8]

$$\begin{aligned} &iU_\xi + U_{\tau\tau} + 2|U|^2U + ic_1U_{\tau\tau\tau} + ic_2(|U|^2U)_\tau \\ &\qquad + ic_3(|U|^4U)_\tau + \gamma|U|^4U = 0, \end{aligned} \tag{1}$$

where $U(\xi, \tau)$ represents a normalized complex amplitude of the pulse envelope, ξ is a normalized distance along the fiber, τ is normalized time with the frame of the reference moving along the fiber at the group velocity, and the coefficients c_i and γ are all real. The terms in (1) are: the group velocity dispersion (GVD), self-phase modulation (SPM): The c_1 and c_2 terms result from including the cubic term in the expansion of the propagation constant and the effects of third order dispersion (TOD) for ultrashort pulses [15, 16], respectively, and the quintic c_3 and γ terms originate from nonlinear dispersion and SPM, respectively. Of course, one can neglect all the c_i terms of (1) for pulse widths greater than 100 fs, and the resulting equation reduces to a NLS equation with cubic-quintic terms, which has been well studied [9, 11]. The HONLS models without γ and c_3 but including the Raman term, i. e. $U(|U|^2)_\tau$, responsible for the self-frequency, a phenomena discovered by Mitschke and Mollenauer [17], have been extensively investigated by various authors [4 - 7].

In the following, we assume a solution of the form

$$\begin{aligned} U(\xi,\tau) &= P(\tau+\beta\xi)\exp[\mathrm{i}(\kappa\xi-\omega\tau)] \\ &= P(\chi)\exp[\mathrm{i}(\kappa\xi-\omega\tau)], \end{aligned} \tag{2}$$

where $P(\chi)$ is a differentiable real function and β is a real constant to be determined by the physical parameters of the model [7, 15]. Substituting (2) into (1) and removing the exponential term, we obtain the real and imaginary parts as

$$\begin{aligned} &(\gamma + c_3\omega)P(\chi)^5 + (2 + c_2\omega)P(\chi)^3 \\ &+ (-\kappa - \omega^2 - c_1\omega^3)P(\chi) + (1 + 3\,c_1\omega)\frac{\mathrm{d}^2}{\mathrm{d}\chi^2}P(\chi) = 0, \end{aligned} \tag{3}$$

$$\begin{aligned} &\left(-2\omega - 3c_1\omega^2 + 3c_2P(\chi)^2 + 5c_3P(\chi)^4 + \beta\right)\frac{\mathrm{d}}{\mathrm{d}\chi}P(\chi) \\ &+ c_1\frac{\mathrm{d}^3}{\mathrm{d}\chi^3}P(\chi) = 0. \end{aligned} \tag{4}$$

By differentiating (3) with respect to χ and comparing with terms in (4), we find that the (3) and (4) are equivalent provided that

$$\frac{\gamma + c_3\omega}{1 + 3\,c_1\omega} = \frac{c_3}{c_1}, \tag{5}$$

$$\frac{2 + c_2\omega}{1 + 3\,c_1\omega} = \frac{c_2}{c_1}, \tag{6}$$

$$-\frac{\kappa + \omega^2 + c_1\omega^3}{1 + 3\,c_1\omega} = \frac{\beta - 2\,\omega - 3\,c_1\omega^2}{c_1}. \tag{7}$$

From (6) and (7), we get

$$\omega = \frac{2\,c_1 - c_2}{2c_1c_2}, \tag{8}$$

$$\kappa = 8\,c_1\omega^3 + 8\,\omega^2 + \frac{(-3\,\beta\,c_1 + 2)\,\omega}{c_1} - \frac{\beta}{c_1}, \tag{9}$$

and the constraint equation for the model coefficients c_3, γ, and c_2 from (5) as

$$\gamma = \frac{2c_3}{c_2}. \tag{10}$$

To find solitary wave solutions, we solve the ordinary differential equation in (3)

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}\chi^2}P(\chi) &= -\frac{\gamma + c_3\omega}{1 + 3\,c_1\omega}P(\chi)^5 - \frac{2 + c_2\omega}{1 + 3\,c_1\omega}P(\chi)^3 \\ &\quad + \frac{\kappa + \omega^2 + c_1\omega^3}{1 + 3\,c_1\omega}P(\chi) \\ &\equiv \alpha_3P(\chi)^5 + \alpha_2P(\chi)^3 + \alpha_1P(\chi), \end{aligned} \tag{11}$$

where

$$\begin{aligned} \alpha_3 &= -\frac{\gamma\,c_2}{2c_1},\ \alpha_2 = -\frac{c_2}{c_1}, \\ \alpha_1 &= -\frac{c_2{}^2 + 4\,c_2{}^2\beta\,c_1 + 4\,c_1c_2 - 12\,c_1{}^2}{c_2{}^2 4c_1{}^2}. \end{aligned} \tag{12}$$

By using the coupled amplitude-phase formulation [14], we can then recast (11) as

$$\mathrm{d}\chi = \left[\alpha_1P^2 + \frac{\alpha_2}{2}P^4 + \frac{\alpha_3}{3}P^6 + 2E\right]^{-1/2}\mathrm{d}P, \tag{13}$$

where E is an arbitrary constant of integration which corresponds to the energy of an harmonic oscillator [15]. This equation is not analytically integrable for non-zero E. In the following, for the case $E = 0$, we will show that the dark solitary wave solution exists under a constraint between c_1 and c_2 in addition to the constraint in (10). Contrary to our result, for the HONLS model with cubic terms [14, 15] the dark solitary and bright solitary wave solutions have been obtained for $E > 0$ and $E = 0$, respectively.

By letting the bracket in (13) be a perfect square, we obtain the integrable differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\chi}P(\chi) = \frac{1}{\eta}\left[P(\chi)^3 + \alpha P(\chi)\right], \tag{14}$$

where $\eta = \sqrt{3/\alpha_3}$, $\alpha = 3\alpha_2/4\alpha_1$, and the constraint $\alpha_2^2 = 16\alpha_1\alpha_3$ which can be expressed using κ, ω, and c_3 in (8) - (10) as

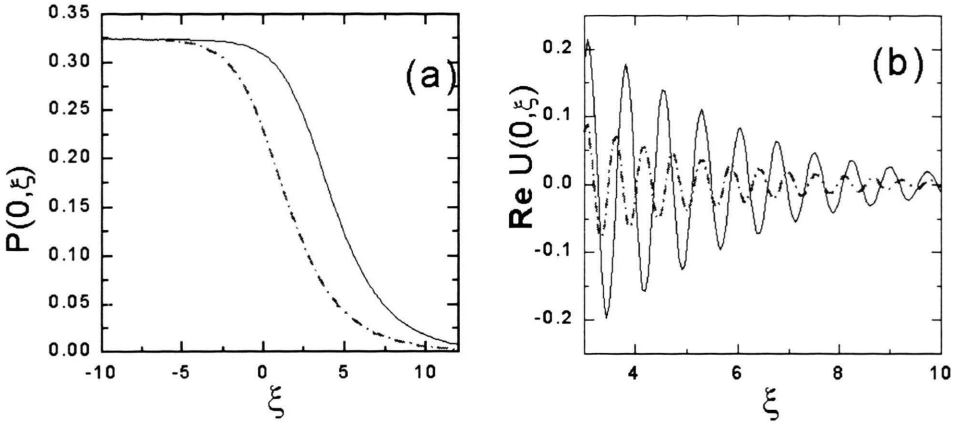

Fig. 1. (a) The dark solitary wave amplitudes $P(0,\xi)$ (solid line) for $C = 1$ and $P_\alpha(0,\xi)$ (dot-dashed line) for $C = 1/\alpha$ with $c_2 = 0.07, c_3 = 0.5, \beta = 700, \gamma = 7.14$, and $c_1 = -3.5 \times 10^{-4}$. (b) Representation of the real part of $U(0,\xi)$ with the same coefficients as in (a).

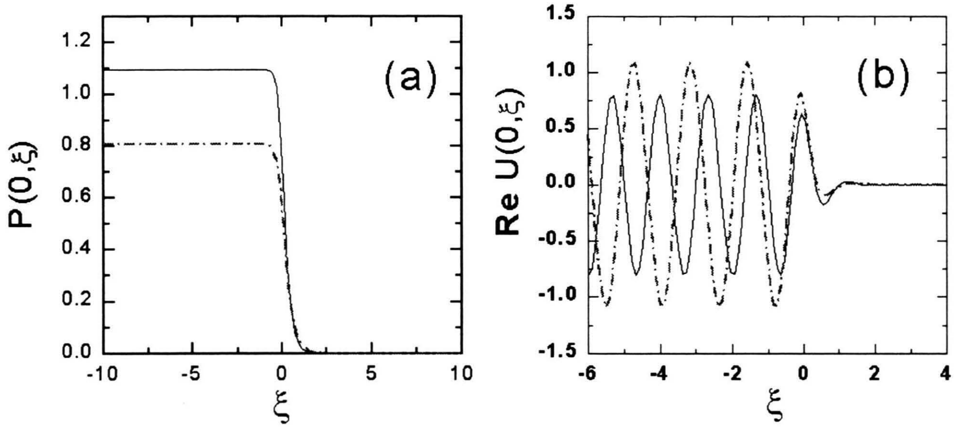

Fig. 2. (a) and (b) The same plots as Figs. 1a and 1b but with $c_2 = -0.8$, $c_3 = -0.5$, $\beta = 10$, $\gamma = 1.25$, and $c_1 = 1.91$.

$$c_1 \equiv \mathbf{c}_1^{\pm} = \frac{{c_2}^2}{6\beta} - \frac{1}{48\gamma}\Big({c_2}^2 - 8\,\gamma \pm \Big[{c_2}^4 - 16\,{c_2}^3\gamma\,\beta - 16\,\gamma\,{c_2}^2 + 64\,\gamma^2\beta^2{c_2}^2 + 128\,\gamma^2\beta\,c_2 + 256\,\gamma^2\Big]^{1/2}\Big)c_2. \quad (15)$$

Then, the solitary wave solution of (14) is

$$P(\chi) = \Big[\frac{\alpha}{1 + C\alpha\exp^{2\alpha\chi/\eta}}\Big]^{-1/2}, \quad (16)$$

where C is an arbitrary constant. We note that for $C = 1/\alpha$ the solution takes the form

$$P_\alpha(\chi) = \sqrt{\frac{\alpha}{2}} \cdot \exp^{-\alpha\chi/2\eta}\cosh^{1/2}\big(\frac{\alpha\chi}{\eta}\big), \quad (17)$$

where $\eta = \sqrt{-6c_1/c_2\gamma}$, $\alpha = 3/2\gamma$, and the soliton power $P_0 = \sqrt{\alpha/2} = \sqrt{3c_2/8c_3}$ (defined as the power at $\chi = 0$, i. e. $\xi = \tau = 0$) depends on the model coefficients c_2 and c_3 or γ. Thus, for zero energy we find the dark solitary wave solution with the modulation term as

$$U(\xi,\tau) = \Big[\frac{3c_3}{2c_2} + C\exp^{\sqrt{3/8}(\tau+\beta\xi)/(-c_1/2c_3^3)^{1/2}}\Big]^{-1/2} \cdot \exp^{i\big[(8\,c_1\omega^3 + 8\,\omega^2 + \frac{(-3\,\beta\,c_1+2)\omega}{c_1} - \frac{\beta}{c_1})\xi - (\frac{2\,c_1 - c_2}{2c_1c_2})\tau\big]}, \quad (18)$$

where $c_1c_3 < 0$ and $c_2c_3 > 0$ should be satisfied along with constraints in (10) and (15).

To explicitly demonstrate the dark solitary wave solutions, we plot 18 by choosing some proper model coefficients for c_i and γ which satisfy the above constraints: $c_2 = 0.07$, $c_3 = 0.5$, and $\beta = 700$ so that $\gamma = 7.14$ and $c_1 = \mathbf{c}_1^- = -3.5 \times 10^{-4}$ (required by the constraint $c_1c_3 < 0$) for Fig. 1; and $c_2 = -0.8, c_3 = -0.5$, and $\beta = 10$ so that $\gamma = 1.25$ and $c_1 = \mathbf{c}_1^+ = 1.91$ (required by the constraint $c_1c_3 < 0$) for Figure 2. The shapes of the amplitude functions for $P(0,\xi)$ ($C = 1$) and $P_\alpha(0,\xi)$ ($C = 1/\alpha$), respectively, which show dark (or kink) solitary wave behavior, are plotted as a function of the normalized distance ξ in Figs. 1a and 2a. The real parts of the dark solitary wave solutions for $U(0,\xi)(C = 1)$ and $U_\alpha(0,\xi)(C = 1/\alpha)$, respectively, which clearly show oscillating and decaying amplitudes, are plotted in Figs. 1b and 2b. We note that similar figures are expected when we plot Figs. 1 and 2 as a function of the normalized time τ.

In conclusion, by using the coupled amplitude-phase formulation [14, 15], we have found the analytical dark solitary wave solutions for the HONLS equation with cubic-quintic terms, which can be important model for femtosecond optical soliton propagation in non-Kerr media. We note that the dark solitary wave solutions exist for the coefficient of quintic term γ much larger than the cubic term compared with the results in [9 - 13]. [9, 10, 11, 12, 13]. Our analysis can not prove the existence of analytical bright solitary wave solutions, however, one may be able to find it via the Painlevé analysis, which will be left for future investigation [18].

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